The laminar and turbulent mixing of jets of compressible fluid. Part II The mixing of two semi-infinite streams

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SUMMARY

This paper presents the application of the methods developed in a previous paper (Part I, Crane & Pack 1957) to the mixing of two parallel streams for both laminar and turbulent flows. The effects of both high velocity and large temperature difference are treated together. The method used consists in developing the stream function in a double series of powers of two parameters, the first being the Mach number and the second depending on the temperature difference of the streams. Analytical expressions are found for the terms up to the second order in the series for the stream function when the streams do not differ too greatly in velocity and temperature. However, when one of the streams is at rest the analytical method is no longer sufficiently accurate, and for this case numerical solutions are given.

For laminar mixing the most important effect is that of 'change of scale', as was found in Part I for a laminar jet at large distances from the orifice. For turbulent half-jets the effect of 'change of scale' and the effect of the perturbation terms due to the Mach number of the flows are approximately equal and opposite, leaving the form of the velocity profile sensibly unchanged from that in incompressible flow. This last result is confirmed by comparison with some experiments of Laurence (1955) on a two-dimensional jet at $M = 0{\cdot}7$. Lastly, the effect of temperature differences is shown to be relatively unimportant even when these are fairly considerable.

INTRODUCTION

The mixing of two semi-infinite incompressible streams has been studied by Görtler (1942), by Lessen (1949), and by Lock (1951). Görtler's method of analysis, applied to turbulent flow in the streams, was adapted by Pai (1954) to the study of laminar mixing of incompressible streams. Görtler's solution is here used as a starting-point for the calculation of the mixing of streams of compressible fluid. The functions originally calculated numerically by Görtler are given below in analytical form. It has been found necessary to treat separately the cases (i) where the velocities of the streams are not very different and (ii) where one of the streams is at rest. This is because the convergence of the series obtained by Görtler's method is not sufficiently rapid in case (ii).

Equations of motion

Let u, v be the velocity components (or mean velocity components in the case of turbulent flow) parallel to cartesian (x, y)-axes. Let the origin of coordinates be taken at the point at which the mixing begins. Let ρ be the density of the gas in the jet, and T the absolute temperature. The assumptions on which the theory rests in this paper are exactly those used in Part I (Crane & Pack 1957) in establishing the equations of motion. These assumptions are that the boundary layer equations hold and that in the turbulent case there exists a coefficient of eddy kinematic viscosity ϵ which is independent of the y coordinate. Hence the basic momentum equation for both laminar and turbulent half-jet mixing is equation (6) of Part I, namely

$$\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial\zeta\partial z}-\frac{\partial\psi}{\partial\zeta}\frac{\partial^2\psi}{\partial z^2}=\alpha\frac{\partial}{\partial z}\left\{T^{*\beta}\frac{\partial^2\psi}{\partial z^2}\right\}. \tag{1}$$

In this equation, ψ is the two-dimensional stream function defined by $\rho u = \partial\psi/\partial y$, $\rho v = -\partial\psi/\partial x$; z is a variable defined by $z = \int_0^y \rho/\rho_0\, dy$; $T^* = T/T_0$ (the suffix 0 refers to a fixed state of the fluid to be defined later); and ζ is a function of x. For laminar flows $\zeta = x$, $\alpha = \mu_0$ and $\beta = n-1$, n coming from the law of variation of the coefficient of viscosity μ with T (i.e. $\mu \propto T^n$). For turbulent flows $\zeta = \int_0^x e(x)\,dx$, where $e(x)$ is an experimentally determined function of x defined by $\epsilon = \epsilon_0 e(x)$, ϵ_0 being a constant; also, $\alpha = \epsilon_0\rho_0$ and $\beta = -2$. Let the two streams in their uniform state have speeds U_1, U_2 parallel to the x-axis, as they cross the half-lines $y > 0$, $y < 0$ respectively. Let

$$U_0 = \tfrac{1}{2}(U_1+U_2), \qquad \lambda = (U_1-U_2)/(U_1+U_2).$$

The solution giving similarity of velocity profiles is expressed by

$$\psi = (\rho_0 U_0\alpha\zeta)^{1/2}g(\eta), \qquad \eta = \left(\frac{U_0\rho_0}{\alpha\zeta}\right)^{1/2} z$$

and the equation for $g(\eta)$ is

$$\frac{d}{d\eta}\{T^{*\beta}g''\}+\tfrac{1}{2}gg'' = 0, \tag{2}$$

where dashes indicate differentiations with respect to η.

As in Part I, if the Prandtl number is assumed to be unity then a particular integral (Crocco's relation) satisfying the boundary layer equations of energy and momentum for both laminar and turbulent flow of jet type is

$$\tfrac{1}{2}u^2+i = A+Bu,$$

where A, B are constants, the values of which depend upon the boundary conditions in the given problem, and $i = C_p T$ (the enthalpy per unit mass). Crocco's relation has to satisfy the conditions $u = U_1$, $T = T_1$, $\rho = \rho_1$ at $y = +\infty$, and $u = U_2$, $T = T_2$, $\rho = \rho_2$ at $y = -\infty$. These lead to the following expression for the Crocco relation:

$$T^* = 1+\frac{h}{\lambda}(u^*-1)-\omega_0^2 u^{*2}, \tag{3}$$

where

$$T^* = T/T_0, \qquad u^* = u/U_0, \qquad \omega_0^2 = U_0^2/2C_pT_0,$$

$$T_0 = \frac{I_1+I_2}{2C_p}, \qquad I_i = C_pT_i+\tfrac{1}{2}U_i^2 \quad (i = 1, 2), \qquad h = \frac{I_1-I_2}{I_1+I_2}.$$

The properties μ_0 and ρ_0 of the gas are taken at the temperature T_0. The quantities I_1 and I_2 are the stagnation enthalpies in the reservoirs from which the uniform streams may be supposed to have come.

The equation (3) shows how the initial discontinuities of temperature and velocity are smoothed out when the mixing is completed. In fact, the stagnation enthalpy at a point in the jet where the velocity is u, obtained by bringing the fluid at that point to rest adiabatically, is equal to

$$I_1+\frac{(I_2-I_1)(u-U_1)}{U_2-U_1}.$$

Let $g(\eta)$ in (2) be expanded in the double series:

$$g(\eta) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{rs}f_{rs}(\xi)\omega_0^{2r}h^s, \tag{4}$$

where $\xi = \frac{1}{2}(\eta-b)$, $\eta = b$ giving the locus of those points of the flow where the velocity is U_0, and all the a_{rs} are equal to $-\beta$ except a_{00} which is equal to 2. Then

$$u^* = \frac{u}{U_0} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \tfrac{1}{2}a_{rs}f'_{rs}(\xi)\omega_0^{2r}h^s$$

and

$$T^{*\beta} = 1+\beta\left\{\frac{h}{\lambda}(u^*-1)-\omega_0^2\,u^*\right\}+\dots. \tag{5}$$

The above expressions are valid for sufficiently small h and ω_0^2, since the numerical value of $(u^*-1)/\lambda$ is less than unity.

The boundary conditions on $g'(\eta)$ are

$$g'(+\infty) = 1+\lambda, \qquad g'(b) = 1, \qquad g'(-\infty) = 1-\lambda.$$

Lock* (1951) has shown for the incompressible case that any given solution of (2) generates an infinity of solutions, all of which satisfy the same boundary conditions at $\eta = \pm\infty$. These solutions may be obtained by replacing η by $\eta+c$ in the given solution, c being an arbitrary constant. This result can be shown to be true for the compressible case. Each one of this infinity of solutions is an equally valid solution of the boundary layer problem because the solution obtained by replacing η by $\eta+c$ in any given solution leads to a value of the y-component of the velocity on the boundary which differs by an amount of order $U_0c/\sqrt{(Re)}$ from that of the given solution, Re being a large dimensionless constant. (In laminar flow Re is the usual Reynolds number; when the flow is turbulent Re is defined as in the laminar case except that the coefficient of kinematic viscosity μ_0/ρ_0 is replaced by the eddy coefficient of kinematic viscosity ϵ_0.) To pick out the correct

* The author is indebted to the referee for drawing his attention to Lock's work, and also for other helpful criticism.

solution it is necessary to take into consideration those terms of the Navier–Stokes equations of order $1/\sqrt{Re}$ higher than the terms of the boundary layer equations. It follows from the above discussion that the velocity at any finite η is mathematically indeterminate from the boundary layer equations alone. Thus b, which by definition gives the locus of points in the flow whose velocity is U_0, is likewise indeterminate. The value of b can however be found from experiment. When b is known the solution to the problem is uniquely determined.

The equations for the functions f_{rs} are obtained by substituting (4) and (5) into (2) and equating the coefficients of $\omega_0^{2r}h_s$ to zero. Thus

$$f_{00}''' + 2f_{00}'' f_{00} = 0, \tag{6}$$

$$f_{10}''' + 2(f_{00} f_{10}'' + f_{00}'' f_{10}) + 2\frac{d}{d\xi}\{f_{00}'^2 f_{00}''\} = 0, \tag{7}$$

$$f_{01}''' + 2(f_{00} f_{01}'' + f_{00}'' f_{01}) - 2\frac{d}{d\xi}\left\{f_{00}''\frac{f_{00}'-1}{\lambda}\right\} = 0, \tag{8}$$

and so on. The boundary conditions are

$$f_{00}'(+\infty) = 1+\lambda, \qquad f_{00}'(0) = 1, \qquad f_{00}'(-\infty) = 1-\lambda,$$
$$f_{10}'(+\infty) = 0, \qquad f_{10}'(0) = 0, \qquad f_{10}'(-\infty) = 0,$$
$$f_{01}'(+\infty) = 0, \qquad f_{01}'(0) = 0, \qquad f_{01}'(-\infty) = 0.$$

The solution of these equations will be found for two separate cases, namely, when λ is small (say less than 0·3) and when $\lambda = 1$, the latter corresponding to one of the streams (here the lower stream) being at rest. In this way many examples of practical interest are likely to be covered.

Case I. Solution for small values of λ

(1) The functions f_{rs} may be expanded in series of ascending powers of λ. If

$$f_{00} = \sum_{n=0}^{\infty} F_n(\xi)\lambda^n,$$

then

$$F_0'(0) = 1, \qquad F_0'(\pm\infty) = 1,$$
$$F_1'(0) = 0, \qquad F_1'(\pm\infty) = \pm 1,$$
$$F_n'(0) = 0, \qquad F_n'(\pm\infty) = 0, \qquad (n = 2, 3, \ldots).$$

The functions F_n are to be derived from differential equations obtained by inserting the above expansion of f_{00} in (6) and equating to zero the coefficients of successive powers of λ. They were first evaluated numerically by Görtler. They may also be expressed in analytical form, and have been tabulated here from these analytical expressions (see table 1). The numerical values show that the expansion of f_{00} does not converge very quickly when λ is greater than about 0·3. For this reason the important case when $\lambda = 1$ is considered separately below.

ξ	F_1 (e)	F_1' (o)	F_2 (o)	F_2' (e)	F_3' (o)	G_1 (e)	G_1' (o)	G_2 (o)	G_2' (e)	H_1 (o)	H_1' (e)	H_2' (o)
0	−0·1610	0	0	0	0	−0·8051	0	0	0	0	0	0
0·1	−0·1554	0·1125	0	0·0018	−0·0024	−0·8107	−0·1117	−0·0007	−0·0215	0·0003	0·0090	−0·0124
0·2	−0·1386	0·2227	0·0004	0·0070	−0·0047	−0·8272	−0·2168	−0·0059	−0·0890	0·0024	0·0347	−0·0220
0·3	−0·1110	0·3286	0·0015	0·0148	−0·0067	−0·8536	−0·3094	−0·0171	−0·1738	0·0077	0·0735	−0·0269
0·4	−0·0731	0·4284	0·0035	0·0244	−0·0084	−0·8885	−0·3846	−0·0398	−0·2808	0·0173	0·1202	−0·0270
0·5	−0·0256	0·5205	0·0064	0·0347	−0·0099	−0·9299	−0·4394	−0·0733	−0·3881	0·0318	0·1688	−0·0236
0·6	0·0307	0·6039	0·0104	0·0443	−0·0114	−0·9757	−0·4723	−0·1169	−0·4807	0·0509	0·2133	−0·0192
0·7	0·0949	0·6778	0·0153	0·0524	−0·0129	−1·0236	−0·4839	−0·1685	−0·5474	0·0741	0·2489	−0·0168
0·8	0·1660	0·7421	0·0208	0·0583	−0·0145	−1·0718	−0·4760	−0·2253	−0·5814	0·1003	0·2726	−0·0186
0·9	0·2430	0·7969	0·0268	0·0614	−0·0162	−1·1183	−0·4518	−0·2837	−0·5815	0·1282	0·2827	−0·0256
1·0	0·3250	0·8427	0·0330	0·0619	−0·0180	−1·1617	−0·4151	−0·3405	−0·5508	0·1564	0·2798	−0·0375
1·1	0·4112	0·8802	0·0391	0·0598	−0·0196	−1·2010	−0·3701	−0·3930	−0·4960	0·1838	0·2655	−0·0527
1·2	0·5008	0·9103	0·0449	0·0557	−0·0209	−1·2356	−0·3208	−0·4392	−0·4254	0·2092	0·2425	−0·0687
1·3	0·5931	0·9340	0·0502	0·0501	−0·0217	−1·2652	−0·2707	−0·4779	−0·3479	0·2321	0·2139	−0·0832
1·4	0·6875	0·9523	0·0549	0·0436	−0·0219	−1·2898	−0·2225	−0·5088	−0·2710	0·2519	0·1826	−0·0940
1·5	0·7834	0·9661	0·0589	0·0368	−0·0213	−1·3098	−0·1784	−0·5323	−0·2004	0·2686	0·1511	−0·0998
1·6	0·8806	0·9763	0·0622	0·0301	−0·0201	−1·3257	−0·1396	−0·5492	−0·1400	0·2822	0·1214	−0·1004
1·7	0·9786	0·9838	0·0650	0·0240	−0·0184	−1·3379	−0·1066	−0·5607	−0·0914	0·2930	0·0949	−0·0962
1·8	1·0777	0·9891	0·0670	0·0186	−0·0163	−1·3472	−0·0796	−0·5670	−0·0546	0·3013	0·0722	−0·0881
1·9	1·1764	0·9928	0·0687	0·0141	−0·0139	−1·3540	−0·0580	−0·5720	−0·0285	0·3076	0·0535	−0·0774
2·0	1·2758	0·9953	0·0699	0·0104	−0·0116	−1·3589	−0·0413	−0·5739	−0·0117	0·3122	0·0387	−0·0655

Table 1. Values of F_1, etc., for small values of λ. (o denotes an odd function of ξ; e denotes an even function of ξ.)

The following are the analytical forms of Görtler's functions, in which

$$\Phi(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\xi^2} d\xi, \qquad \Phi_1(\xi) = \frac{2}{\sqrt{\pi}} e^{-\xi^2},$$

$$\beta_1 = -\frac{\sqrt{\pi}}{2}\left(\frac{1}{2}+\frac{1}{\pi}\right) \doteqdot -0{\cdot}72521.$$

$$F_0 = \xi^*,$$

$$F_1 = \xi\Phi + \tfrac{1}{2}\Phi_1 + \beta_1,$$

$$F_2 = -\tfrac{1}{2}\xi\Phi^2 - \tfrac{3}{4}\Phi\Phi_1 + \beta_1\Phi + \sqrt{(2/\pi)}\Phi(\sqrt{2}\xi) + \tfrac{1}{2}\xi,$$

$$\begin{aligned} F_3' = {} & \tfrac{1}{2}\Phi^3 - (\tfrac{1}{4}\xi^3 + \tfrac{7}{8}\xi)\Phi^2\Phi_1 - (\tfrac{1}{4}\xi^2 + \tfrac{3}{4})\Phi_1^2\Phi - \tfrac{1}{16}\xi\Phi_1^3 - (3\sqrt{3}/4\pi)\Phi(\sqrt{3}\xi) - \\ & - \beta_1(\xi^2 + \tfrac{1}{2})\Phi\Phi_1 - \tfrac{1}{2}\beta_1\xi\Phi_1^2 + (3\sqrt{3}/4\pi - \tfrac{1}{2})\Phi + \xi(\tfrac{1}{4} - \beta_1^2)\Phi_1 + \\ & + \sqrt{(2/\pi)}\Phi_1\Phi(\sqrt{2}\xi). \end{aligned}$$

(F_3 is not evaluated since $F_3(0) = d_3$ which is unknown.)

(2) Next let

$$f_{10} = \sum_0^{\infty} G_n(\xi)\lambda^n.$$

The boundary conditions are

$$G_n'(0) = G_n'(\pm\infty) = 0, \quad (n = 0, 1, \ldots).$$

The equations for G_n obtained by insertion into (7) are

$$G_0''' + 2\xi G_0'' = 0,$$

$$G_1''' + 2\xi G_1'' + 2G_0 F_1'' + 2F_1''' = 0,$$

and so on. The solution to the first of these equations satisfying the boundary conditions is found to be $G_0 = a_0$, where a_0 is a constant to be determined. In the equation for G_1, put $G_1'' = \Phi_1\, y(\xi)$. Then

$$y' + 2a_0 - 4\xi = 0.$$

Thus $$G_1'' = (2\xi^2 - 2a_0\xi + C)\Phi_1,$$

where C is a constant. On integration and application of the boundary conditions, it is found that $C = -1$, $a_0 = 0$, and that

$$G_1 = \tfrac{1}{2}\Phi_1 + a_1.$$

The value of a_1 is found when the solution for G_2' is calculated. In fact,

$$a_1 = \sqrt{\pi}\left(\frac{3}{2\pi} - \frac{5}{4}\right),$$

$$G_2' = -2\Phi^2 + \Phi\Phi_1(\xi^3 - \tfrac{3}{2}\xi) + \Phi_1^2(\tfrac{1}{2}\xi^2 - 1) + (2\beta_1\xi^2 - \beta_1 + a_1)\Phi_1 + 2$$

where β_1 has the same value as before. The analytical forms of the functions G_0, G_1, G_2 are therefore

$$G_0 = 0,$$

$$G_1 = \tfrac{1}{2}\Phi_1 + a_1, \quad \text{with } a_1 = \sqrt{\pi}\left(\frac{3}{2\pi} - \frac{5}{4}\right) = -1{\cdot}36927,$$

$$G_2 = -2\xi\Phi^2 - \left(\frac{1}{2}\xi^2 + \frac{7}{4}\right)\Phi\Phi_1 + \sqrt{\left(\frac{2}{\pi}\right)}\,\Phi(\sqrt{2}\xi) - \tfrac{1}{4}\xi\Phi_1^2 - \beta_1\xi\Phi_1 + a_1\Phi + 2\xi.$$

* The solution satisfying the boundary conditions on F_0 is $F_0 = \xi + c$, where c is a constant. On insertion of this expression for F_0 in the equation for F_1, it is seen that $c = 0$ if F_1 is to satisfy the boundary conditions.

The expressions for G_3, G_4, etc., could be found in the same way, but the labour involved in equating the complicated analytical terms would be very great, for example, G_3 alone involves about 60 terms. When the above expressions for G_1' and G_2' are used, f_{10}' is obtained to 1% accuracy for values of λ up to about 0·2 if the Mach number is less than five.

(3) The same form of expansion is used for f_{01}, namely

$$f_{01} = \sum_0^\infty H_n(\xi)\lambda^n,$$

with the boundary conditions

$$H_n'(0) = H_n'(\pm\infty) = 0 \quad (n = 0, 1, \ldots).$$

A procedure similar to that used above then yields the results

$$\begin{aligned} H_0 &= c_1 = \tfrac{1}{2}\sqrt{\pi}(1-2/\pi) = 0{\cdot}32204, \\ H_1 &= c_1\Phi + \xi\Phi^2 + \tfrac{1}{2}\Phi\Phi_1 - \xi, \\ H_2' &= \{(3\beta_1 - \tfrac{1}{2}c_1) - \xi^2(c_1 + 2\beta_1)\}\Phi\Phi_1 - \xi(\tfrac{1}{2}c_1 + \beta_1)\Phi_1^2 - (\tfrac{4}{3})\Phi^3 + \\ &\quad + (\tfrac{1}{2}\xi - \xi^3)\Phi^2\Phi_1 - \xi^2\Phi_1^2\Phi + (\sqrt{3}/\pi)\Phi(\sqrt{3}\xi) - \\ &\quad - \tfrac{1}{4}\xi\Phi_1^3 + (\tfrac{4}{3} - \sqrt{3}/\pi)\Phi - 2\beta_1 c_1 \xi\Phi_1. \end{aligned}$$

(H_2 has not been determined.)

As for the other functions, the expressions for the higher H-functions could be found exactly, but the labour would be very great. However, when only the functions up to H_2 are used, f_{01}' is found quite accurately for values of λ up to about 0·3. Indeed, on comparing $H_1' + H_2'$ with the exact solution for f_{01}' when $\lambda = 1$, it is seen that in the range $|\xi| < 1{\cdot}5$ the agreement is quite good. The maximum error in this range is about 5%.

Case II. Solution for $\lambda = 1$

When $\lambda = 1$ the lower stream is at rest. The solutions of (6) to (8) are required subject to the conditions

$$f_{00}'(+\infty) = 2, \qquad f_{00}'(0) = 1, \qquad f_{00}'(-\infty) = 0,$$

with all the other conditions unchanged.

An iterative method was used to solve equation (6). The solution tabulated below is in agreement with that of Lock (1951) for the same problem.

The next equation to solve is (7). Again a numerical method is required. It is seen that an integral belonging to the complementary function is $f_{10} = f_{00}'$. By the use of this integral it is possible to reduce (7) to a second-order differential equation. When the asymptotic form of the complementary function of (7) is examined it is found that (7) has solutions which result in f_{10}' remaining finite at $\xi = \pm\infty$ whereas the reduced (second-order) equation has integrals which diverge exponentially at $\xi = \pm\infty$. Thus it is advantageous to solve (7) itself numerically, rather than this reduced equation.

Two linearly independent parts of the complementary function of (7) with boundary conditions

$$f_{10}(0) = 0, \qquad f'_{10}(0) = 0, \qquad f''_{10}(0) = 1,$$

and

$$f_{10}(0) = 1, \qquad f'_{10}(0) = 0, \qquad f''_{10}(0) = 0,$$

respectively, were computed. A particular integral of the complete equation (7) with boundary conditions

$$f_{10}(0) = 0, \qquad f'_{10}(0) = 0, \qquad f''_{10}(0) = 0,$$

was obtained. A linear combination of the first two solutions was added to the particular integral to give the solution satisfying the boundary conditions. This treatment of (7) is equally applicable to (8) and was used to find the values of $f_{01}(\xi)$. Values of f_{00}, f_{01}, f_{10} and their first derivatives are given in table 2.

The effect of the change of scale

If ω_0^2 and h are both small,

$$\xi_1 \doteqdot \xi + \frac{h}{\lambda}\int_0^{\xi}(f'_{00}-1)\,d\xi - \omega_0^2\int_0^{\xi}(f'_{00})^2\,d\xi \tag{9}$$

where

$$\xi_1 = \tfrac{1}{2}\left(\frac{U_0\rho_0}{\alpha\zeta}\right)^{1/2} y.$$

When λ is small, this approximation becomes

$$\xi_1 \doteqdot \xi + h[\{F_1 - F_1(0)\} + \lambda F_2 + \lambda^2\{F_3 - F_3(0)\} + \ldots] - \\ - \omega_0^2[\xi + 2\lambda\{F_1 - F_1(0)\} + \ldots],$$

but when $\lambda = 1$ the equation (9) should be used.

These relations show how changes in the width of the mixing region due to the 'change of scale'* depend on the difference of the stagnation enthalpies of the streams and on the Mach numbers of the flow. To a first approximation these effects are independent. By considering $\int_0^{\xi}(f'_{00}-1)\,d\xi$ it can be shown that the *net* effect on the width of the mixing region of differing stagnation enthalpies is very slight when $|h| < 0{\cdot}3$ for any λ. An example of $h \doteqdot 0{\cdot}3$ is afforded by two streams with stagnation temperatures of 300° C and room temperature respectively. The effect of a non-zero Mach number is to decrease the width of the mixing region as the Mach number increases.

Conclusion

The effect of compressibility on the non-dimensional velocity profile may be conveniently divided into two parts. One is the effect of change

* The 'change of scale' effect was defined in Part I as the difference between (i) the velocity profile obtained from the first term in the expansion for ψ, when z is expressed in terms of the physical ordinate y and (ii) the velocity profile in incompressible flow (which comes from this same term with $z = y$).

ξ	f_{00} $(+\xi)$	f'_{00} $(+\xi)$	f_{00} $(-\xi)$	f'_{00} $(-\xi)$	f_{01} $(+\xi)$	f'_{01} $(+\xi)$	f_{01} $(-\xi)$	f'_{01} $(-\xi)$	f_{10} $(+\xi)$	f'_{10} $(+\xi)$	f_{10} $(-\xi)$	f'_{10} $(-\xi)$
0	−0·169	1·000	−0·169	1·000	0·286	0	0·286	0	−0·863	0	−0·863	0
0·1	−0·064	1·112	−0·264	0·892	0·285	−0·006	0·284	0·023	−0·866	−0·084	−0·866	0·045
0·2	0·053	1·224	−0·348	0·790	0·285	0·006	0·280	0·060	−0·880	−0·209	−0·870	0·058
0·3	0·181	1·336	−0·422	0·695	0·287	0·037	0·272	0·105	−0·908	−0·370	−0·875	0·047
0·4	0·320	1·444	−0·487	0·607	0·293	0·081	0·259	0·153	−0·953	−0·555	−0·878	0·020
0·5	0·470	1·545	−0·544	0·527	0·304	0·132	0·242	0·200	−1·007	−0·741	−0·877	−0·011
0·6	0·629	1·637	−0·593	0·456	0·319	0·180	0·219	0·242	−1·098	−0·905	−0·873	−0·043
0·7	0·797	1·718	−0·635	0·392	0·339	0·219	0·193	0·277	−1·194	−1·023	−0·865	−0·071
0·8	0·972	1·787	−0·672	0·337	0·362	0·242	0·164	0·303	−1·298	−1·077	−0·855	−0·093
0·9	1·154	1·844	−0·703	0·288	0·387	0·247	0·133	0·321	−1·404	−1·062	−0·842	−0·110
1·0	1·340	1·889	−0·729	0·245	0·411	0·235	0·101	0·331	−1·505	−0·983	−0·828	−0·120
1·1	1·531	1·924	−0·752	0·209	0·444	0·209	0·067	0·333	−1·595	−0·857	−0·814	−0·124
1·2	1·725	1·949	−0·771	0·177	0·453	0·174	0·034	0·330	−1·671	−0·705	−0·798	−0·123
1·3	1·921	1·968	−0·788	0·150	0·468	0·138	0·002	0·321	−1·732	−0·548	−0·783	−0·119
1·4	2·118	1·980	−0·801	0·127	0·480	0·102	−0·030	0·308	−1·757	−0·401	−0·769	−0·111
1·5	2·316	1·988	−0·813	0·107	0·489	0·072	−0·060	0·293	−1·774	−0·276	−0·754	−0·102
1·6	2·516	1·993	−0·823	0·090	0·495	0·048	−0·088	0·278	−1·794	−0·178	−0·742	−0·091
1·7	2·715	1·996	−0·831	0·076	0·499	0·030	−0·115	0·257	−1·805	−0·100	−0·730	−0·080
1·8	2·915	1·998	−0·838	0·064	0·501	0·018	−0·140	0·239	−1·811	−0·055	−0·719	−0·068
1·9	3·115	1·999	−0·844	0·054	0·502	0·009	−0·162	0·220	−1·813	−0·021	−0·708	−0·057
2·0	3·315	2·000	−0·849	0·045	0·503	0·004	−0·184	0·202	−1·813	−0·008	−0·698	−0·046

Table 2. Values of $f_{00}, f'_{00}, f_{01}, f'_{01}, f_{10}, f'_{10}$, when $\lambda = 1$ (i.e. the lower stream is at rest).

of scale. The other is the change in the velocity brought about by the perturbation terms f'_{01}, f'_{10}, ... The non-dimensional velocity profile is given by

$$\frac{u}{U_0} = f'_{00} + (-\tfrac{1}{2}\beta)hf'_{01} + (-\tfrac{1}{2}\beta)\omega_0^2 f'_{10} + \dots.$$

The perturbation terms in this equation depend on both compressibility and the variation of viscosity with temperature for laminar flow. For laminar flows the effect of change of scale is dominant, the perturbation effect being negligible. As an example, for $\lambda = 1$, the change in the velocity profile produced by putting $\mu \propto T^{0\cdot 76}$ instead of $\mu \propto T$ is at most 4% for Mach numbers up to 5. Furthermore, since for $|h| < 0\cdot 3$ the effect of h on the change of scale is negligible, the basic cause of changes in the profile in compressible laminar flow is the Mach number operating through the change of scale. The larger the Mach number the narrower is the mixing region. For example, when the lower stream is at rest and the upper stream has a speed given by $M = 5$, the width of the mixing region is about three-quarters of that for incompressible flow, the Reynolds numbers being the same. The greater part of the contraction of the profile is in that part where the velocity is highest—as it is expected on physical grounds, since the effect of compressibility is naturally less in the more slowly moving parts of the mixing region. When the temperature of the upper stream is higher than that of the lower, that is, $h > 0$, the upper half of the profile tends to broaden and the lower to shrink. The effect is of course reversed when $h < 0$. This is also to be expected on physical grounds, since molecules in a hotter stream have higher random velocities than those in a cooler one and hence tend to transfer some of the momentum of the faster moving parts of the jet to the surrounding parts to a greater extent than in the cooler stream.

For turbulent flows the perturbation terms and the change of scale have effects of the same order of magnitude. It is found on computing the non-dimensional velocity profile that they almost cancel each other. Thus the form of the non-dimensional velocity profile is left sensibly unchanged from that obtained in incompressible flow. In order to test this last result against experiment a scale factor σ must be used*, where $\xi = \sigma(y-y_0)/x$ if the coefficient of eddy viscosity is taken (following Görtler 1942) to be $\epsilon_0 x/L$. $y = y_0$ gives the locus of points in the flow at which $U = U_0$. From the equations developed above, $\sigma = (U_0 L/2\epsilon_0)^{1/2}$. The theory has been applied to the case examined experimentally by Laurence (1955), namely to a turbulent jet at Mach number 0·7 entering a medium at rest. The observations chosen for comparison were those taken in the neighbourhood of the orifice where the core of the jet was at constant velocity. This might be expected to give the closest representation to the mixing of two parallel streams. The results have been plotted in figure 1 with u/U_c

* The scale factor σ is not an absolute constant but may vary with the Mach number of the undisturbed stream, that is with ω_0^2. This does not affect the statement made in the preceding sentence.

as ordinate, U_c being the velocity of the core of the jet, and with $(y_0 - y)/x$ as abscissa. The best fit between theory and experiment was obtained for $\sigma = 12{\cdot}7$, when $\xi = \sigma(y_0 - y)/x$. The fit is seen to be good except in the part of the profile where the velocity is lower; in this region the gradient of the theoretical profile is less steep than that of the experimental profile.

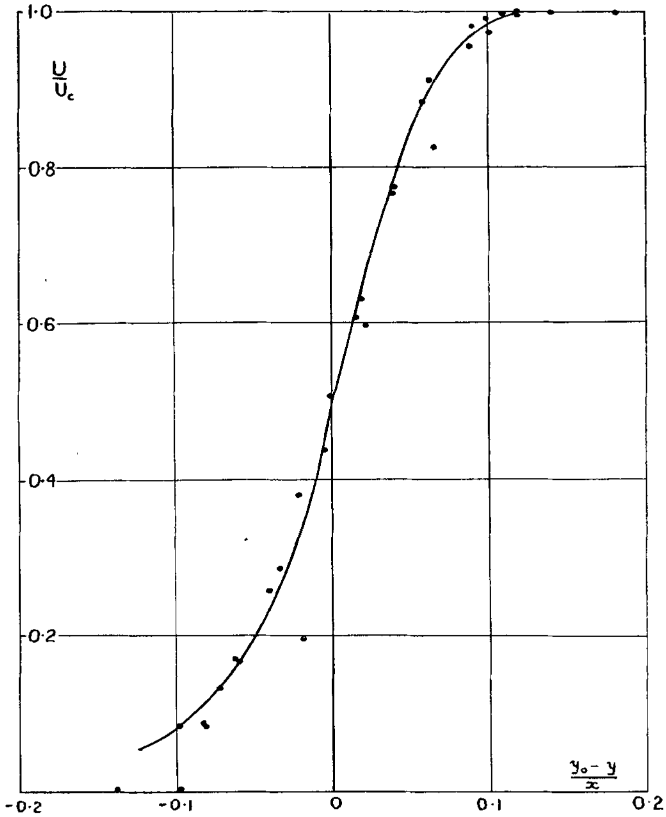

Figure 1. Comparison between Laurence's results and the theoretical profile with $\sigma = 12{\cdot}7$, for the mixing region near the orifice when a turbulent jet of air issues at Mach number 0·7 into air at rest.

Now over the whole flow the gradient of the laminar profile corresponding to the laminar Reynolds number of the jet is steeper than the turbulent profile. This suggests that turbulence may not be fully developed in the lower velocity region, that is, the flow may be only intermittently turbulent there*.

* Subsequent to the writing of this paper the author's attention was drawn to the work of Johannesen (1957). Johannesen has shown that the solution f'_{00} gives an accurate fit, except in the region of lower velocity, to his experimental results for the non-dimensional velocity profile of the mixing region near the orifice when a round turbulent jet issues with Mach number 1·6 into a medium at rest. The value of σ used was 21·9.

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